Spin-dependent point potentials in one and three dimensions

2007 J. Phys. A: Math. Theor. 40 249

(http://iopscience.iop.org/1751-8121/40/2/004)

Institute of Physics Publishing | Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **40** (2007) 249–261 doi:10.1088/1751-8113/40/2/004

Spin-dependent point potentials in one and three dimensions

Claudio Cacciapuoti[1], **Raffaele Carlone**[2] **and Rodolfo Figari**[2]

[1] Institut für Angewandte Mathematik der Universität Bonn. Wegelerstrasse 6, D-53115 Bonn, Germany

[2] Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Napoli. Dipartimento di Scienze Fisiche, Università di Napoli Federico II. Via Cintia 80126 Napoli, Italy

E-mail: caccia@na.infn.it, carlone@na.infn.it and figari@na.infn.it

Received 3 October 2006, in final form 12 November 2006
Published 12 December 2006
Online at stacks.iop.org/JPhysA/40/249

Abstract

We consider a system realized with one spinless quantum particle and an array of N spins $1/2$ in dimensions 1 and 3. We characterize all the Hamiltonians obtained as point perturbations of assigned free dynamics in terms of some generalized boundary conditions. For every boundary condition, we give the explicit formula for the resolvent of the corresponding Hamiltonian. We discuss the problem of locality and give two examples of spin-dependent point potentials that could be of interest as multi-component solvable models.

PACS numbers: 03.65.Nk, 03.65.Yz

1. Introduction

Point interactions were introduced in the early days of quantum mechanics in order to describe the low energy dynamics of a quantum particle subject to short-range forces, see, e.g., [9, 12, 17, 27]. The appearance of divergent terms in a formal perturbation scheme using delta-like potentials was often bypassed considering only the first term in the expansion. Methods and results of the application of this kind of potentials to the theory of neutron scattering by solids and fluids can be found in [18].

The work of Berezin and Faddeev [8] at the beginning of the sixties opened the way to a complete characterization of point interaction Hamiltonians in any dimension (for an exhaustive review of what is currently known about these kind of solvable models, see e.g. [4]). Few years later, Minlos and Faddeev [20] were the first to point out the difficulties to extend zero-range interactions to systems of more than two particles. As an aside, we want to mention that neither a definite way-out of this ultraviolet problem in non-relativistic quantum mechanics nor a no-go theorem has been found yet. For this reason, the range of applicability of point interactions remained limited to the framework of one-particle quantum mechanics.

Nowadays there is a growing interest in multi-component quantum systems and in particular in the study of the dynamics of a microscopic quantum system in interaction with a quantum environment. The evolution of the entanglement system environment and the onset of the transition to a more classical behaviour of the microscopic system as a consequence of the interaction with the environment are the dynamical features under analysis.

In the following, making use of recent techniques in the theory of self-adjoint extensions of symmetric operators, we construct models for the dynamics of one quantum particle in interaction with any number of localized spins. In this way we are able to define simple, but genuinely multi-component, quantum systems where conjectures and qualitative results in the theory of quantum open systems can, in principle, be rigorously approached.

For the sake of simplicity, we examine systems consisting of one spinless particle in interaction with localized $1/2$ spins (in units where $\hbar = 1$). Physical phenomenology would suggest considering the particle with spin and a spin–spin interaction conserving the total spin. It is easy to convince oneself that, in the latter case, inside each channel characterized by a fixed value of the total spin, the dynamics would be described by some Hamiltonian of the type we consider here, possibly relative to a value of the spin larger than $1/2$. Few examples of such Hamiltonians were already heuristically found and used to study different problems, e.g., the spin-dependent scattering [18] or the interaction of one quantum particle with one or (several) quantum dots [7] (see also [19] for one example in two dimensions). The straightforward generalization to higher values of the spin will not be given here.

In section 2, we introduce some notation and define the free quantum dynamics for the particle and the spins. In section 3, we state and prove our main results: we give a complete characterization of all zero-range perturbations of the free dynamics in dimensions 1 and 3. At the end of section 3, we discuss with more detail two examples of spin-dependent point interactions that, in our opinion, are of interest as non-trivial solvable models. In order to make clearer our formulae, the resolvent in the simple case of $N = 1$ and $d = 3$ is written in an extended form. A section of conclusions follows.

2. Some notation and the free dynamics

In this section, we define the state space for a quantum system consisting of one particle and an array of N spins. Moreover we introduce some notation and define the non-interacting Hamiltonian H.

We will consider here the case of spin $1/2$. The state of each spin placed in a fixed position of space is represented by a unitary vector in $\mathbb{C}^2$.

Consider the first Pauli matrix, $\hat{\sigma}_j^{(1)}$, where the index $j = 1, \dots, N$ indicates that such an operator refers to the jth spin. We indicate with χ_{σ_j} the normalized eigenvector of the operator $\hat{\sigma}_j^{(1)}$ with eigenvalue $\sigma_j = \pm 1$

$$\hat{\sigma}_j^{(1)} \chi_{\sigma_j} = \sigma_j \chi_{\sigma_j} \qquad \sigma_j = \pm 1; \quad \left\| \chi_{\sigma_j} \right\|_{\mathbb{C}^2} = 1, \qquad j = 1, \dots, N. \tag{1}$$

With this notation the state of the jth spin can be written as the linear superposition $a_j \chi_+ + b_j \chi_-$, with $a_j, b_j \in \mathbb{C}$ and $|a_j|^2 + |b_j|^2 = 1$.

The natural Hilbert space for the description of a system of one particle in dimension d and N spins $1/2$ is then

$$\mathcal{H} = L^2(\mathbb{R}^d) \otimes \mathbb{S}_N, \tag{2}$$

where

$$\mathbb{S}_N = \overbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}^{N}. \tag{3}$$

In this paper we will consider only the cases $d = 1, 3$. We indicate with a capital Greek letter a generic vector in $\mathcal{H}$.

Let us define $\mathcal{X}_{\underline{\sigma}} = \chi_{\sigma_1} \otimes \cdots \otimes \chi_{\sigma_N}$, where $\underline{\sigma}$ is the N-dimensional vector $\underline{\sigma} = (\sigma_1, \ldots, \sigma_N)$. Trivially $\mathcal{X}_{\underline{\sigma}} \in \mathbb{S}_N$, $\|\mathcal{X}_{\underline{\sigma}}\|_{\mathbb{S}_N} = 1$ and the following decomposition formula holds:

$$\Psi = \sum_{\underline{\sigma}} \psi_{\underline{\sigma}} \otimes \mathcal{X}_{\underline{\sigma}} \qquad \Psi \in \mathcal{H}, \tag{4}$$

where the sum runs over all the possible configurations of the vector $\underline{\sigma}$ while $\psi_{\underline{\sigma}} \in L^2(\mathbb{R}^d)\, \forall \underline{\sigma}$ is referred to as the wavefunction component of the state Ψ. The choice of $\mathcal{X}_{\underline{\sigma}}$ as basis of $\mathbb{S}_N$ is arbitrary; we consider the basis of eigenvectors of $\hat{\sigma}_j^{(1)}$ according to what will be our choice for the free Hamiltonian.

The scalar product in $\mathcal{H}$ is defined in a natural way by

$$\langle \Psi, \Phi \rangle = \sum_{\underline{\sigma}} (\psi_{\underline{\sigma}}, \phi_{\underline{\sigma}})_{L^2} \qquad \Psi, \Phi \in \mathcal{H}. \tag{5}$$

Consider the operator in $\mathbb{S}_N$

$$\mathbf{S}_j = \overbrace{\mathbb{I}_{\mathbb{C}^2} \otimes \cdots \otimes \hat{\sigma}_j^{(1)} \otimes \cdots \otimes \mathbb{I}_{\mathbb{C}^2}}^{N} \qquad j = 1, \ldots, N. \tag{6}$$

Vectors $\mathcal{X}_{\underline{\sigma}}$ are eigenvectors of $\mathbf{S}_j$:

$$\mathbf{S}_j \mathcal{X}_{\underline{\sigma}} = \sigma_j \mathcal{X}_{\underline{\sigma}} \qquad j = 1, \ldots, N. \tag{7}$$

The following operator is self-adjoint in $\mathcal{H}$:

$$D(H) = H^2(\mathbb{R}^d) \otimes \mathbb{S}_N \tag{8}$$

$$H = -\frac{\hbar^2}{2m} \Delta \otimes \mathbb{I}_{\mathbb{S}_N} + \sum_{j=1}^{N} \mathbb{I}_{L^2} \otimes \alpha_j \mathbf{S}_j \qquad \alpha_j \in \mathbb{R}. \tag{9}$$

Here $H^2(\mathbb{R}^d)$ indicates the standard Sobolev space of functions in $L^2(\mathbb{R}^d)$, with first and second generalized derivatives in $L^2(\mathbb{R}^d)$. m indicates the mass of the particle and α_j are real constants with the dimension of an energy. The operator H defines the free Hamiltonian. In the following, we will fix $\hbar = 1$ and $2m = 1$.

By using the decomposition formula (4), it is easily seen that the action of H on vectors in its domain is given by

$$H\Psi = \sum_{\underline{\sigma}} (-\Delta + \underline{\alpha}\, \underline{\sigma}) \psi_{\underline{\sigma}} \otimes \mathcal{X}_{\underline{\sigma}} \qquad \Psi \in \mathcal{H}, \tag{10}$$

where $\underline{\alpha}$ is the N-dimensional real vector $(\alpha_1, \ldots, \alpha_N)$ and $\underline{\alpha}\, \underline{\sigma} = \sum_{j=1}^N \alpha_j \sigma_j$.

The resolvent of H, $R(z) = (H - z)^{-1}$, is

$$R(z)\Psi = \sum_{\underline{\sigma}} (-\Delta - z + \underline{\alpha}\, \underline{\sigma})^{-1} \psi_{\underline{\sigma}} \otimes \mathcal{X}_{\underline{\sigma}} \qquad \Psi \in \mathcal{H}; \quad z \in \rho(H), \tag{11}$$

where $\rho(H)$ indicates the resolvent set of H. We indicate with $G^w(x - x')$ the integral kernel of the operator $(-\Delta - w)^{-1}$. Its explicit expression is well known and reads

$$G^w(x) = \begin{cases} \mathrm{i}\dfrac{\mathrm{e}^{\mathrm{i}\sqrt{w}|x|}}{2\sqrt{w}} & d = 1 \\[2ex] \dfrac{\mathrm{e}^{\mathrm{i}\sqrt{w}|x|}}{4\pi |x|} & d = 3 \end{cases} \qquad w \in \mathbb{C}\backslash\mathbb{R}^+; \quad \mathrm{Im}(\sqrt{w}) > 0. \tag{12}$$

From the spectral properties of the operator $-\Delta$, with domain $D(-\Delta) = H^2(\mathbb{R}^d)$, it is easily seen that the spectrum of H is only absolutely continuous, in particular

$$\sigma_{pp}(H) = \varnothing; \quad \sigma_{ess}(H) = \sigma_{ac}(H) = [\mu, \infty), \qquad \mu = \min_{\underline{\sigma}}(\underline{\alpha}\,\underline{\sigma}). \tag{13}$$

The solution of the Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Psi^t = H\Psi^t, \tag{14}$$

with initial datum

$$\Psi^{t=0} = \Psi^0 = \sum_{\underline{\sigma}} \psi^0_{\underline{\sigma}} \otimes \mathcal{X}_{\underline{\sigma}} \qquad \Psi^0 \in \mathcal{H}, \tag{15}$$

is formally written as $\mathrm{e}^{-\mathrm{i}tH}\Psi_0$. By using the property of the Laplace transform $\mathcal{L}^{-1}(\mathcal{L}(f)(\cdot + s))(\tau) = \mathrm{e}^{\prime s\tau} f(\tau)$, we obtain the strongly continuous unitary group $\mathrm{e}^{-\mathrm{i}tH}$ (see, e.g., Th. VIII.7 [24])

$$\Psi^t = \mathrm{e}^{-\mathrm{i}Ht}\Psi^0 = \sum_{\underline{\sigma}} U^t \psi^0_{\underline{\sigma}} \otimes \mathrm{e}^{-\mathrm{i}\underline{\alpha}\,\underline{\sigma}t}\mathcal{X}_{\underline{\sigma}}, \tag{16}$$

where $U^t : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is the generator of the free dynamics for one particle in d dimensions

$$(U^t f)(x) = \frac{1}{(4\pi \mathrm{i}t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\frac{|x-x'|^2}{4t}} f(x')\, \mathrm{d}x'. \tag{17}$$

The Hamiltonian H does not give rise to any interaction among the particle and the spins and of the spins among themselves.

3. Point perturbations of *H*

In this section we use the theory of self-adjoint extensions of symmetric operators to derive the whole family of Hamiltonians that coincide with H on functions whose support does not contain the set of points where the spins are placed (for an introduction to the standard von Neumann's theory of self-adjoint extensions of symmetric operators, see e.g. [3, 25]).

Let us indicate with Y the set $\{y_1, \dots, y_N\}$, where $y_j \in \mathbb{R}^d$ indicates the position of the jth spin $1/2$. Consider the symmetric operator on $\mathcal{H}$

$$D(H_0) = C_0^\infty(\mathbb{R}^d \backslash Y) \otimes \mathbb{S}_N \tag{18}$$

$$H_0 = -\Delta \otimes \mathbb{I}_{\mathbb{S}_N} + \sum_{j=1}^N \mathbb{I}_{L^2} \otimes \alpha_j \mathbf{S}_j \qquad \alpha_j \in \mathbb{R}. \tag{19}$$

Let $\mathcal{K}_z(H_0) = \mathrm{Ker}[H_0^* - z]$ with $\mathrm{Im}(z) \neq 0$, where * indicates the adjoint. To evaluate the deficiency indices of H_0, $n_+(H_0) = \dim[\mathcal{K}_i]$ and $n_-(H_0) = \dim[\mathcal{K}_{-i}]$, we have to find all the independent solutions of the equation

$$(H_0^* - z)\Phi^z = 0 \qquad z \in \mathbb{C}\backslash\mathbb{R}; \quad \Phi^z \in D(H_0^*). \tag{20}$$

Define $\Phi^z = \sum_{\underline{\sigma}} \phi^z_{\underline{\sigma}} \otimes \mathcal{X}_{\underline{\sigma}}$, then equation (20) is equivalent to

$$\left(\phi^z_{\underline{\sigma}}, (-\Delta - \bar{z} + \underline{\alpha}\,\underline{\sigma})\psi\right)_{L^2} = 0 \qquad \phi^z_{\underline{\sigma}} \in L^2(\mathbb{R}^d); \quad \forall \psi \in C_0^\infty(\mathbb{R}^d \backslash Y); \quad z \in \mathbb{C}\backslash\mathbb{R}. \tag{21}$$

The independent solutions of (20) in $\mathcal{H}$ are

$$\begin{cases} \Phi^z_{0j\underline{\sigma}} = G^{z-\underline{\alpha}\,\underline{\sigma}}(\cdot - y_j) \otimes \mathcal{X}_{\underline{\sigma}} \\ \Phi^z_{1j\underline{\sigma}} = (G^{z-\underline{\alpha}\,\underline{\sigma}})'(\cdot - y_j) \otimes \mathcal{X}_{\underline{\sigma}} \end{cases} \qquad z \in \mathbb{C}\backslash\mathbb{R} \qquad d = 1 \tag{22}$$

$$\Phi^z_{j\underline{\sigma}} = G^{z-\underline{\alpha}\,\underline{\sigma}}(\cdot - y_j) \otimes \mathcal{X}_{\underline{\sigma}} \qquad z \in \mathbb{C}\backslash\mathbb{R} \qquad d = 3, \tag{23}$$

where $G^w(x)$, $w \in \mathbb{C}\backslash\mathbb{R}^+$, is defined in (12).

$(G^w)'$ indicates the first derivative of G^w with respect to x

$$(G^w)'(x) = -\mathrm{sgn}(x)\frac{\mathrm{e}^{\mathrm{i}\sqrt{w}|x|}}{2} \qquad w \in \mathbb{C}\backslash\mathbb{R}^+; \quad \mathrm{Im}(\sqrt{w}) > 0 \qquad d = 1. \tag{24}$$

Since the index $\underline{\sigma}$ runs over 2^N distinct configurations and $j = 1, \ldots, N$, for $d = 1$ the deficiency indices are $n_+ = n_- = N2^{N+1}$ while for $d = 3$ one has $n_+ = n_- = N2^N$. Von Neumann's theory ensures that self-adjoint extensions of H_0 exist and they are parametrized by the unitary applications between $\mathcal{K}_i$ and $\mathcal{K}_{-i}$. Accordingly, the family of operators which are self-adjoint extensions of H_0 is characterized by $(N2^{N+1})^2$ real parameters for $d = 1$ and by $(N2^N)^2$ real parameters for $d = 3$.

Let us denote with $H^{\mathcal{U}}$ the self-adjoint extension of H_0 corresponding, via the von Neumann's formula, to the unitary application $\mathcal{U} : K_i(H_0) \to K_{-i}(H_0)$. In general, given $\mathcal{U}$, it is not easy to obtain any information about the resolvent of $H^{\mathcal{U}}$ and the behaviour of the wavefunction component of the generic vector $\Psi \in D(H^{\mathcal{U}})$ in the points y_j.

Since we want to stress the relation between a given self-adjoint operator and the coupling between the wavefunction and the spin placed in y_j, we characterize the self-adjoint extensions in terms of some generalized boundary conditions satisfied by the wavefunction component of the vector Ψ.

As was shown in [13] there is a one-to-one correspondence between the self-adjoint extensions of a given symmetric operator H_0 and the self-adjoint linear relations on $\mathbb{C}^m$, where $m = n_+(H_0) = n_-(H_0)$. Moreover, in [6] (see also [22]) it was shown, in a very general setting, that a generalized Krein's formula for the resolvent exists. Such a formula explicitly gives the resolvent of a self-adjoint extension of a given symmetric operator in terms of the parameters characterizing the boundary conditions satisfied by the vectors in its domain. Moreover the generalized formula for the resolvent given in [6, 22] avoids the problem of finding the maximal common part of two extensions.

In this paper we use the results of [6, 22] to obtain a complete characterization in terms of generalized boundary conditions of all the self-adjoint extensions of the operator H_0. Moreover we explicitly give a formula for the resolvent of each self-adjoint extension of H_0.

Let use introduce the following notation. With μ we indicate the multi-index $\mu = (pj\underline{\sigma})$ for $d = 1$ and $\mu = (j\underline{\sigma})$ for $d = 3$. Indices p, p', p'', etc always assume the values 0 and 1. Indices j, j' and so on run over $1, \ldots, N$. With $\underline{\sigma}$, $\underline{\sigma}'$, etc, we indicate N-dimensional vectors, e.g., $(\sigma_1, \ldots, \sigma_N)$ where $\sigma_j = \pm 1$. As an example with this notation, the vectors in $\mathcal{H}$ defined by (22) and (23) are shortly referred to as Φ^z_μ.

In the following, $\delta_{i,j}$ indicates the Kronecker symbol

$$\delta_{i,j} = \begin{cases} 1 & i = j \\ 0 & i \neq j; \end{cases} \tag{25}$$

moreover

$$\delta_{\underline{\sigma},\underline{\sigma}'} = \delta_{\sigma_1,\sigma_1'} \ldots \delta_{\sigma_N,\sigma_N'}. \tag{26}$$

Given two $m \times m$ matrices A and B, $(A|B)$ indicates the $m \times 2m$ block matrix with the first m columns given by the columns of A and the second m's given by the columns of B.

Theorem 1. ($d = 1$) *Define the operator*

$$D(H^{AB}) = \Bigg\{ \Psi = \sum_{\underline{\sigma}} \psi_{\underline{\sigma}} \otimes \mathcal{X}_{\underline{\sigma}} \in \mathcal{H} | \psi_{\underline{\sigma}} \in H^2(\mathbb{R} \backslash Y)\, \forall\, \underline{\sigma};$$

$$\sum_{\mu'} A_{\mu,\mu'} q_{\mu'} = \sum_{\mu'} B_{\mu,\mu'} f_{\mu'}; \tag{27}$$

$$q_{0j\underline{\sigma}} = \psi'_{\underline{\sigma}}(y_j^-) - \psi'_{\underline{\sigma}}(y_j^+), \qquad q_{1j\underline{\sigma}} = \psi_{\underline{\sigma}}(y_j^-) - \psi_{\underline{\sigma}}(y_j^+), \tag{28}$$

$$f_{pj\underline{\sigma}} = (-)^p \frac{\psi_{\underline{\sigma}}^{(p)}(y_j^+) + \psi_{\underline{\sigma}}^{(p)}(y_j^-)}{2}, \tag{29}$$

$$AB^* = BA^*,\ (A|B) \text{ of maximal rank } N2^{N+1} \Bigg\} \tag{30}$$

$$H^{AB}\Psi = \sum_{\underline{\sigma}} (-\Delta + \underline{\alpha}\,\underline{\sigma}) \psi_{\underline{\sigma}} \otimes \mathcal{X}_{\underline{\sigma}} \qquad \alpha_j \in \mathbb{R}, \qquad x \in \mathbb{R} \backslash Y. \tag{31}$$

H^{AB} is self-adjoint and its resolvent, $R^{AB}(z) = (H^{AB} - z)^{-1}$, is given by

$$R^{AB}(z) = R(z) + \sum_{\mu,\mu',\mu''} ((\Gamma^{AB}(z))^{-1})_{\mu,\mu'} B_{\mu',\mu''} \langle \Phi^{\bar{z}}_{\mu''}, \cdot \rangle \Phi^z_\mu \qquad z \in \rho(H^{AB}), \tag{32}$$

where $\Gamma^{AB}(z)$ is the $N2^{N+1} \times N2^{N+1}$ matrix defined as

$$\Gamma^{AB}(z) = B\Gamma(z) + A \tag{33}$$

with

$$\begin{aligned}
&(\Gamma(z))_{pj\underline{\sigma},p'j'\underline{\sigma}'} = 0 && \underline{\sigma} \neq \underline{\sigma}' \\
&(\Gamma(z))_{pj\underline{\sigma},p'j\underline{\sigma}} = 0 && p \neq p' \\
&(\Gamma(z))_{0j\underline{\sigma},0j'\underline{\sigma}} = -G^{z-\underline{\alpha}\,\underline{\sigma}}(y_j - y_{j'}) && \\
&(\Gamma(z))_{1j\underline{\sigma},1j'\underline{\sigma}} = -(z - \underline{\alpha}\,\underline{\sigma}) G^{z-\underline{\alpha}\,\underline{\sigma}}(y_j - y_{j'}) && \\
&(\Gamma(z))_{1j\underline{\sigma},0j'\underline{\sigma}} = (G^{z-\underline{\alpha}\,\underline{\sigma}})'(y_j - y_{j'}) && j \neq j' \\
&(\Gamma(z))_{0j\underline{\sigma},1j'\underline{\sigma}} = -(G^{z-\underline{\alpha}\,\underline{\sigma}})'(y_j - y_{j'}) && j \neq j'.
\end{aligned} \tag{34}$$

Functions $G^w(x)$ and $(G^w)'(x)$ are defined in (12) and (24), respectively.

Proof. Define two linear applications $\Lambda : D(H_0^*) \to \mathbb{C}^m$ and $\tilde{\Lambda} : D(H_0^*) \to \mathbb{C}^m$, with $m = N2^{N+1}$. Λ defines the charges q_μ in (28) by

$$q_\mu = (\Lambda\Psi)_\mu \qquad \mu = (pj\underline{\sigma}); \qquad \Psi = \sum_{\underline{\sigma}} \psi_{\underline{\sigma}} \otimes \mathcal{X}_{\underline{\sigma}} \in D(H_0^*). \tag{35}$$

$\tilde{\Lambda}$ defines f_μ in (29)

$$f_\mu = (\tilde{\Lambda}\Psi)_\mu \qquad \mu = (pj\underline{\sigma}); \qquad \Psi = \sum_{\underline{\sigma}} \psi_{\underline{\sigma}} \otimes \mathcal{X}_{\underline{\sigma}} \in \mathcal{D}(\mathcal{H}_\prime^*). \tag{36}$$

The linear functionals Λ and $\tilde{\Lambda}$ correspond to Γ_1 and Γ_2 defined in [6]. Integrating by parts, it follows that

$$\langle \Psi_1, H_0^*\Psi_2 \rangle - \langle H_0^*\Psi_1, \Psi_2 \rangle = \sum_\mu [\overline{(\Lambda\Psi_1)}_\mu (\tilde{\Lambda}\Psi_2)_\mu - \overline{(\tilde{\Lambda}\Psi_1)}_\mu (\Lambda\Psi_2)_\mu] \tag{37}$$

for all $\Psi_1, \Psi_2 \in D(H_0^*)$. Moreover Λ and $\tilde{\Lambda}$ are surjective; this implies that the triple $(\mathbb{C}^m, \Lambda, \tilde{\Lambda})$ is a boundary value space for H_0, see e.g. [13]. Then from theorem 3.1.6 in [13] we obtain that all the self-adjoint extensions of H_0 correspond to the restrictions of H_0^* on vectors Ψ satisfying

$$\sum_{\mu'} A_{\mu,\mu'}(\Lambda\Psi)_{\mu'} = \sum_{\mu'} B_{\mu,\mu'}(\tilde{\Lambda}\Psi)_{\mu'}, \tag{38}$$

where $A_{\mu,\mu'}$ and $B_{\mu,\mu'}$ are two $N2^{N+1}$ matrices satisfying $AB^* = BA^*$ (AB^* Hermitian) and $(A|B)$ with maximal rank $N2^{N+1}$. This proves that the operators H^{AB} are self-adjoint.

We use the proposition proved in [6] (see also theorem 10 in [22]) to write down the resolvent of H^{AB}.

Define $\gamma_z : \mathbb{C}^m \to \mathcal{K}_z$ in the following way: $\gamma_z = (\Lambda|\mathcal{K}_z)^{-1}$. The action of γ_z on a vector $\underline{a} \in \mathbb{C}^m$ is given by

$$\gamma_z \underline{a} = \sum_{\mu} a_\mu \Phi^z_\mu, \tag{39}$$

where Φ^z_μ is defined in (22). In fact,

$$\left(\Lambda\Phi^z_{p'j'\underline{\sigma}'}\right)_{pj\underline{\sigma}} = \delta_{\underline{\sigma},\underline{\sigma}'}\delta_{j,j'}\delta_{p,p'}[(G^{z-\underline{\alpha}\,\underline{\sigma}})'(0^-) - (G^{z-\underline{\alpha}\,\underline{\sigma}})'(0^+)] = \delta_{\underline{\sigma},\underline{\sigma}'}\delta_{j,j'}\delta_{p,p'}. \tag{40}$$

The adjoint of γ_z, $\gamma_z^* : \mathcal{H} \to \mathbb{C}^m$ is defined by

$$(\gamma_z^*\Psi)_\mu = \left\langle \Phi^z_\mu, \Psi \right\rangle; \tag{41}$$

in fact

$$\langle \Psi, \gamma_z \underline{a}\rangle = \sum_{pj\underline{\sigma}} a_{pj\underline{\sigma}} (\psi_{\underline{\sigma}}(\cdot), (G^{z-\underline{\alpha}\,\underline{\sigma}})^{(p)}(\cdot - y_j))_{L^2} = \sum_{pj\underline{\sigma}} \overline{(\gamma_z^*\Psi)}_{pj\underline{\sigma}} a_{pj\underline{\sigma}}. \tag{42}$$

By straightforward calculations, it is possible to show that the matrix $\Gamma(z) = -\tilde{\Lambda}\gamma_z$ coincides with the definition given in (34). From the definition of the domain of H^{AB}, it follows that the free Hamiltonian H is the self-adjoint extension of H_0 corresponding to the choice $A = 1$ and $B = 0$. Then γ_z and $\Gamma(z)$ are analytic for $z \in \rho(H)$, and

$$(\Gamma(z))_{\mu,\mu'} - (\Gamma(w))_{\mu,\mu'} = (w - z)\left\langle \Phi^{\bar{z}}_\mu, \Phi^w_{\mu'} \right\rangle \qquad z, w \in \rho(H). \tag{43}$$

Making use of the result stated in [6] (see also theorem 10 in [22]), we obtain that for all $z \in \rho(H) \cap \rho(H^{AB})$ the resolvent formula (32) holds. Since the resolvent of H^{AB} is a finite rank perturbation of the resolvent of H, we have $\sigma_{ess}(H^{AB}) = \sigma_{ess}(H) = \sigma(H)$ (see, e.g., [5]), and $\rho(H) \cap \rho(H^{AB}) = \rho(H^{AB})$. □

An analogous theorem holds in the three-dimensional case.

Theorem 2. ($d = 3$) *Define the operator*

$$D(H^{AB}) = \Bigg\{ \Psi = \sum_{\underline{\sigma}} \psi_{\underline{\sigma}} \otimes \mathcal{X}_{\underline{\sigma}} \in \mathcal{H} | \Psi = \Psi^z + \sum_\mu q_\mu \Phi^z_\mu;$$

$$\Psi^z \in D(H);\ z \in \rho(H^{AB});$$

$$\sum_{\mu'} A_{\mu,\mu'} q_{\mu'} = \sum_{\mu'} B_{\mu,\mu'} f_{\mu'}; \tag{44}$$

$$q_{j\underline{\sigma}} = \lim_{|x-y_j|\to 0} 4\pi|x - y_j|\psi_{\underline{\sigma}}(x), \tag{45}$$

$$f_{j\underline{\sigma}} = \lim_{|x-y_j|\to 0} \left[\psi_{\underline{\sigma}}(x) - \frac{q_{j\underline{\sigma}}}{4\pi|x - y_j|}\right], \tag{46}$$

$$AB^* = BA^*,\ (A|B)\ \textit{of maximal rank}\ N2^N \Bigg\} \tag{47}$$

$$H^{AB}\Psi = H\Psi^z + z\sum_{j,\underline{\sigma}} q_{j\underline{\sigma}}\Phi^z_{j\underline{\sigma}} \qquad \Psi \in D(H^{AB}). \tag{48}$$

H^{AB} is self-adjoint and its resolvent, $R^{AB}(z) = (H^{AB} - z)^{-1}$, is given by

$$R^{AB}(z) = R(z) + \sum_{\mu,\mu',\mu''} ((\Gamma^{AB}(z))^{-1})_{\mu,\mu'} B_{\mu',\mu''}\langle\Phi^{\bar{z}}_{\mu''}, \cdot\rangle\Phi^z_\mu \qquad z \in \rho(H^{AB}), \tag{49}$$

where $\Gamma^{AB}(z)$ is the $N2^N \times N2^N$ matrix defined as

$$\Gamma^{AB}(z) = B\Gamma(z) + A. \tag{50}$$

with

$$\begin{aligned}
&(\Gamma(z))_{j\underline{\sigma},j'\underline{\sigma}'} = 0 \qquad \underline{\sigma} \neq \underline{\sigma}' \\
&(\Gamma(z))_{j\underline{\sigma},j\underline{\sigma}} = \frac{\sqrt{z - \underline{\alpha}\,\underline{\sigma}}}{4\pi\mathrm{i}} \\
&(\Gamma(z))_{j\underline{\sigma},j'\underline{\sigma}} = -G^{z-\underline{\alpha}\,\underline{\sigma}}(y_j - y_{j'}) \qquad j \neq j'.
\end{aligned} \tag{51}$$

The function $G^w(x)$ is defined in (12).

Proof. The proof of the self-adjointness of H^{AB} is basically the same as in the one-dimensional case. Two linear, surjective applications $\Lambda,\ \tilde{\Lambda} : D(H_0^*) \to \mathbb{C}^m$, define the charges $q_{j\underline{\sigma}}$ and the values $f_{j\underline{\sigma}}$ as was done in the one-dimensional case, see (35) and (36). The von Neumann decomposition formula (see, e.g., [25]) gives the following expression for the generic vector in $D(H_0^*)$:

$$\Psi = \Psi_0 + \sum_\mu \left(a_\mu\Phi^i_\mu + b_\mu\Phi^{-i}_\mu\right) \qquad a_\mu, b_\mu \in \mathbb{C};\ \Psi^0 \in D(H_0) \tag{52}$$

with $\Phi^{\pm i}_\mu$ as in (23). The action of H_0^* on its domain can be written as

$$H_0^*\Psi = H_0\Psi_0 + \mathrm{i}\sum_\mu \left(a_\mu\Phi^i_\mu - b_\mu\Phi^{-i}_\mu\right) \qquad a_\mu, b_\mu \in \mathbb{C};\ \Psi_0 \in D(H_0). \tag{53}$$

By using the symmetry of H_0 it is easily proved that, given $\Psi_1, \Psi_2 \in D(H_0^*)$ such that

$$\Psi_k = \Psi_{k,0} + \sum_\mu \left(a_{k,\mu}\Phi^i_\mu + b_{k,\mu}\Phi^{-i}_\mu\right) \qquad a_{k,\mu}, b_{k,\mu} \in \mathbb{C};\ \Psi_{k,0} \in D(H_0), \quad k = 1, 2, \tag{54}$$

the following relation holds:

$$\begin{aligned}
&\langle\Psi_1, H_0^*\Psi_2\rangle - \langle H_0^*\Psi_1, \Psi_2\rangle \\
&\quad = 2\mathrm{i}\sum_{j,j',\underline{\sigma}} (\bar{a}_{1,j\underline{\sigma}}a_{2,j'\underline{\sigma}} - \bar{b}_{1,j\underline{\sigma}}b_{2,j'\underline{\sigma}})(G^{i-\underline{\alpha}\,\underline{\sigma}}(\cdot - y_j), G^{i-\underline{\alpha}\,\underline{\sigma}}(\cdot - y_{j'}))_{L^2}.
\end{aligned} \tag{55}$$

On the other hand,

$$(\Lambda\Psi_k)_\mu = q_{k,\mu} = a_{k,\mu} + b_{k,\mu}, \qquad k = 1, 2 \tag{56}$$

and

$$\begin{aligned}
(\tilde{\Lambda}\Psi_k)_{j\underline{\sigma}} = f_{k,j\underline{\sigma}} &= \mathrm{i}\left(a_{k,j\underline{\sigma}}\frac{\sqrt{\mathrm{i} - \underline{\alpha}\,\underline{\sigma}}}{4\pi} + b_{k,j\underline{\sigma}}\frac{\sqrt{-\mathrm{i} - \underline{\alpha}\,\underline{\sigma}}}{4\pi}\right) \\
&\quad + \sum_{j'\neq j}(a_{k,j'\underline{\sigma}}G^{i-\underline{\alpha}\,\underline{\sigma}}(y_j - y_{j'}) + b_{k,j'\underline{\sigma}}G^{-i-\underline{\alpha}\,\underline{\sigma}}(y_j - y_{j'})), \qquad k = 1, 2.
\end{aligned} \tag{57}$$

The right-hand side of relation (37) then reads

$$
\begin{aligned}
&\sum_\mu [\overline{(\Lambda\Psi_1)_\mu}(\tilde{\Lambda}\Psi_2)_\mu - \overline{(\tilde{\Lambda}\Psi_1)_\mu}(\Lambda\Psi_2)_\mu] \\
&\qquad = \frac{\mathrm{i}}{4\pi}(\bar{a}_{1,j\underline{\sigma}}a_{2,j\underline{\sigma}} - \bar{b}_{1,j\underline{\sigma}}b_{2,j\underline{\sigma}})(\sqrt{\mathrm{i} - \underline{\alpha}\,\underline{\sigma}} - \sqrt{-\mathrm{i} - \underline{\alpha}\,\underline{\sigma}}) \\
&\qquad\quad + \sum_{j'\neq j}(\bar{a}_{1,j\underline{\sigma}}a_{2,j'\underline{\sigma}} - \bar{b}_{1,j\underline{\sigma}}b_{2,j'\underline{\sigma}})(G^{i-\underline{\alpha}\,\underline{\sigma}}(y_j - y_{j'}) - G^{-i-\underline{\alpha}\,\underline{\sigma}}(y_j - y_{j'})). \qquad (58)
\end{aligned}
$$

By using the resolvent identity on $(G^{i-\underline{\alpha}\,\underline{\sigma}}(\cdot - y_j), G^{i-\underline{\alpha}\,\underline{\sigma}}(\cdot - y_{j'}))_{L^2}$, for $j \neq j'$, and by direct computation of $\|G^{i-\underline{\alpha}\,\underline{\sigma}}\|^2_{L^2}$, it is shown that (55) and (58) coincide. Then, also for $d = 3$, the triple $(\mathbb{C}^m, \Lambda, \tilde{\Lambda})$ is a boundary value space and the restriction of H_0^* to vectors satisfying (44) is self-adjoint; we indicate such a restriction with $\tilde{H}^{AB}$. Assume that $\Psi \in \tilde{H}^{AB}$ and that it is written as in formula (52), posing

$$
\Psi = \Psi^z + \sum_\mu q_\mu \Phi^z_\mu \qquad (59)
$$

with

$$
\Psi^z = \Psi_0 + \sum_\mu \left(a_\mu \Phi^i_\mu + b_\mu \Phi^{-i}_\mu - q_\mu \Phi^z\right), \qquad (60)
$$

and noting that $q_\mu = a_\mu + b_\mu$, it follows that $\Psi^z \in D(H)$ and that the action of $\tilde{H}^{AB}$ on its domain is given by (48). Then H^{AB} is self-adjoint.

Define $\gamma_z : \mathbb{C}^m \to \mathcal{K}_z$ as before: $\gamma_z = (\Lambda|\mathcal{K}_z)^{-1}$. Analogous to the one-dimensional case, given a vector $\underline{a} \in \mathbb{C}^m$, $\gamma_z\underline{a} = \sum_\mu a_\mu \Phi^z_\mu$ (see theorem 3). Its adjoint is $\gamma_z^* : \mathcal{H} \to \mathbb{C}^m$, $(\gamma_z^*\Psi)_\mu = \langle \Phi^z_\mu, \Psi \rangle$. As in the one-dimensional case, it is possible to show that the matrix $\Gamma(z) = -\tilde{\Lambda}\gamma_z$ coincides with the definition given in (51). The free Hamiltonian H corresponds to the choice $A = 1$ and $B = 0$, and the resolvent formula (49) follows as in the one-dimensional case. □

If the matrix B is invertible, the generalized Krein formula is easily reduced to the standard formula with one matrix usually denoted with Θ, see [23].

The generalized boundary conditions of forms (27) and (44) include both local and non-local interactions. In our setting, local means that the behaviour of the wavefunction in the point y_j depends only on the state of the spin placed in the point y_j. The sub-family of local Hamiltonians H^{AB}, the only ones generally considered physically admissible, is obtained by imposing some restrictions on the matrices A and B, i.e.

$d = 1$

$$
\begin{array}{lll}
A_{pj\underline{\sigma},p'j'\underline{\sigma}'} = B_{pj\underline{\sigma},p'j'\underline{\sigma}'} = 0 & \forall\, j \neq j' & \\
A_{pj\underline{\sigma},p'j\underline{\sigma}'} = B_{pj\underline{\sigma},p'j\underline{\sigma}'} = 0 & \text{if for some } k \neq j, \quad \sigma_k \neq \sigma'_k & \\
A_{pj\underline{\sigma},p'j\underline{\sigma}'} = a_{pj\sigma_j,p'j\sigma'_j}; & B_{pj\underline{\sigma},p'j\underline{\sigma}'} = b_{pj\sigma_j,p'j\sigma'_j} & \text{otherwise}
\end{array} \qquad (61)
$$

$d = 3$

$$
\begin{array}{lll}
A_{j\underline{\sigma},j'\underline{\sigma}'} = B_{j\underline{\sigma},j'\underline{\sigma}'} = 0 & \forall\, j \neq j' & \\
A_{j\underline{\sigma},j\underline{\sigma}'} = B_{j\underline{\sigma},j\underline{\sigma}'} = 0 & \text{if for some } k \neq j, \quad \sigma_k \neq \sigma'_k & \\
A_{j\underline{\sigma},j\underline{\sigma}'} = a_{j\sigma_j,j\sigma'_j}; & B_{j\underline{\sigma},j\underline{\sigma}'} = b_{j\sigma_j,j\sigma'_j} & \text{otherwise,}
\end{array} \qquad (62)
$$

where the (complex) constants $a_{pj\sigma_j,p'j\sigma'_j}$, $b_{pj\sigma_j,p'j\sigma'_j}$ (and $a_{j\sigma_j,j\sigma'_j}$, $b_{j\sigma_j,j\sigma'_j}$) are subjected to restriction (30) (and (47)).

We give the explicit form of two local Hamiltonians that we consider of special interest.

Example 1. δ-like interactions.

Consider the following choice for the matrices A and B:

$$
\begin{array}{ll}
d=1 & d=3 \\
a_{pj\sigma_j,p'j\sigma_j'}=\delta_{p,p'}\delta_{\sigma_j,\sigma_j'} & a_{j\sigma_j,j\sigma_j'}=\beta_{j\sigma_j}\delta_{\sigma_j,\sigma_j'} \\
b_{0j\sigma_j,0j\sigma_j'}=-2\beta_{j\sigma_j}\delta_{\sigma_j,\sigma_j'} & b_{j\sigma_j,j\sigma_j'}=\delta_{\sigma_j,\sigma_j'} \\
b_{pj\sigma_j,p'j\sigma_j'}=0 \quad \text{for } p\neq 0 \text{ or } p'\neq 0 & \text{with } \beta_{j\sigma_j}\in\mathbb{R} \\
\text{with } \beta_{j\sigma_j}\in\mathbb{R}. &
\end{array}
\tag{63}
$$

We indicate with H^δ the generic Hamiltonian in this sub-family of local interactions. For $d=1$, the wavefunction component of the generic state $\Psi \in D(H^\delta)$ is continuous but with discontinuous derivative, in particular the following boundary conditions hold:

$$\psi_{\underline{\sigma}}\big(y_j^+\big)=\psi_{\underline{\sigma}}(y_j^-)\equiv\psi_{\underline{\sigma}}(y_j), \qquad \psi'_{\underline{\sigma}}\big(y_j^+\big)-\psi'_{\underline{\sigma}}(y_j^-)=\beta_{j\sigma_j}\psi_{\underline{\sigma}}(y_j). \tag{64}$$

For $d=3$ the boundary conditions simply read

$$\beta_{j\sigma_j}q_{j\underline{\sigma}}=f_{j\underline{\sigma}}. \tag{65}$$

Following a practice common in the literature (see [4] and references therein), we refer to H^δ as δ-like interactions. We would like to stress that such boundary conditions are diagonal in the spin variables. This means that the χ_+ component of the jth spin affects only the wavefunction component relative to the configuration of the spins with the jth one in the state χ_+. This implies that, given the initial state $\Psi^{t=0}=\psi^0\otimes\mathcal{X}_{\underline{\sigma}}$, the evolution generated by H^δ gives $\Psi^t=\psi^t\otimes\mathcal{X}_{\underline{\sigma}}$. Here $\psi^t(x)=\big(U_\beta^t\psi^0\big)(x)$, where U_β^t is a strongly continuous unitary group in $L^2(\mathbb{R}^d)$.

An analogous remark holds for all the boundary conditions that are diagonal in the spin variables. While in dimension 3 they are only of the form given in the example, in dimension 1 the family of self-adjoint boundary conditions is richer. Among them we recall the ones corresponding to a δ' coupling (see [4]), whose domain consists of discontinuous wavefunctions with continuous derivative such that the jump of the wavefunction in y_j is proportional to the value of the first derivative in y_j.

Example 2. Off-diagonal interactions.

Let us consider the local interactions defined by

$$
\begin{array}{ll}
d=1 & d=3 \\
a_{pj\sigma_j,p'j\sigma_j'}=\delta_{p,p'}\delta_{\sigma_j,\sigma_j'} & a_{j\sigma_j,j\sigma_j'}=\sigma_j i\hat{\beta}_{j\sigma_j}\big(1-\delta_{\sigma_j,\sigma_j'}\big) \\
b_{0j\sigma_j,0j\sigma_j'}=-2\sigma_j i\hat{\beta}_{j\sigma_j}\big(1-\delta_{\sigma_j,\sigma_j'}\big) & b_{j\sigma_j,j\sigma_j'}=\delta_{\sigma_j,\sigma_j'} \\
b_{pj\sigma_j,p'j\sigma_j'}=0 \quad \text{for } p\neq 0 \text{ or } p'\neq 0 & \text{with } \hat{\beta}_{j\sigma_j}\in\mathbb{R} \\
\text{with } \hat{\beta}_{j\sigma_j}\in\mathbb{R}. &
\end{array}
\tag{66}
$$

A simple calculation gives the corresponding boundary conditions. For $d=1$

$$
\begin{aligned}
&\psi_{\underline{\sigma}}\big(y_j^+\big)=\psi_{\underline{\sigma}}(y_j^-)\equiv\psi_{\underline{\sigma}}(y_j) \\
&\psi'_{\underline{\sigma}}\big(y_j^+\big)-\psi'_{\underline{\sigma}}(y_j^-)=\sigma_j i\beta_{j\sigma_j}\psi_{(\sigma_1\ldots\sigma_j'\ldots\sigma_N)}(y_j) \qquad \sigma_j'\neq\sigma_j,
\end{aligned}
\tag{67}
$$

and for $d=3$

$$\sigma_j i\hat{\beta}_{j\sigma_j}q_{j(\sigma_1\ldots\sigma_j'\ldots\sigma_N)}=f_{j\underline{\sigma}} \qquad \sigma_j'\neq\sigma_j. \tag{68}$$

The class of Hamiltonians proposed in this second example are the simplest off-diagonal ones. The interaction with the particle induces the spins to evolve towards a superposition state also when the initial state is such that every spin is in an eigenstate of $\hat{\sigma}_j^{(1)}$, $\Psi^{t=0} = \psi^0 \otimes \mathcal{X}_{\underline{\sigma}}$.

We regard as useful to give, at least in the simplest case of one spin, the explicit expression of the resolvent of the Hamiltonians proposed in examples 1 and 2. This is done in the following.

Example 3. One spin in dimension 3.

Let us consider the case of one spin in dimension 3 placed in the point $y \in \mathbb{R}^3$. We indicate with $R^\delta(z)$ the resolvent of the Hamiltonian H^δ defined in example 1 when $N = 1$. The resolvent $R^\delta(z)$ can be written as

$$R^\delta(z) = \left[G^{z-\alpha} + \frac{4\pi\mathrm{i}}{\sqrt{z-\alpha}+4\pi\mathrm{i}\beta_+} G^{z-\alpha}(\cdot - y)G^{z-\alpha}(y - \cdot)\right] \otimes (\chi_+, \cdot)_{\mathbb{C}^2}\chi_+ \\ + \left[G^{z+\alpha} + \frac{4\pi\mathrm{i}}{\sqrt{z+\alpha}+4\pi\mathrm{i}\beta_-} G^{z+\alpha}(\cdot - y)G^{z+\alpha}(y - \cdot)\right] \otimes (\chi_-, \cdot)_{\mathbb{C}^2}\chi_-. \tag{69}$$

The expressions in the square brackets are identical to the resolvent of the operator formally written as '$-\Delta + \beta_\sigma\delta_y$' in dimension 3 (see [4]). Then all the results concerning the delta potential in dimension 3 can be adapted to H^δ. Let us recall that the generator of the dynamics can be formally written as $\mathrm{e}^{-\mathrm{i}H^\delta t} = -\mathcal{L}^{-1}((H^\delta - \cdot)^{-1})(-\mathrm{i}t)$, then due to the presence of the projectors $(\chi_+, \cdot)_{\mathbb{C}^2}\chi_+$ and $(\chi_-, \cdot)_{\mathbb{C}^2}\chi_-$ the dynamics generated by H^δ is factorized in the spin components.

Let us indicate with H^{od} the Hamiltonian corresponding to the one defined in example 2, in dimension 3 and with $N = 1$. Its resolvent can be explicitly written with the following large formula:

$$\begin{aligned} R^{od}(z) = {} & G^{z-\alpha} \otimes (\chi_+, \cdot)_{\mathbb{C}^2}\chi_+ + G^{z+\alpha} \otimes (\chi_-, \cdot)_{\mathbb{C}^2}\chi_- \\ & - \frac{4\pi\mathrm{i}\sqrt{z+\alpha}}{(4\pi)^2\hat{\beta}_+\hat{\beta}_- \sqrt{z-\alpha}\sqrt{z+\alpha}} G^{z-\alpha}(\cdot - y)G^{z-\alpha}(y - \cdot) \otimes (\chi_+, \cdot)_{\mathbb{C}^2}\chi_+ \\ & - \frac{4\pi\mathrm{i}\sqrt{z-\alpha}}{(4\pi)^2\hat{\beta}_+\hat{\beta}_- \sqrt{z-\alpha}\sqrt{z+\alpha}} G^{z+\alpha}(\cdot - y)G^{z+\alpha}(y - \cdot) \otimes (\chi_-, \cdot)_{\mathbb{C}^2}\chi_- \\ & - \frac{\mathrm{i}\hat{\beta}_+}{(4\pi)^2\hat{\beta}_+\hat{\beta}_- \sqrt{z-\alpha}\sqrt{z+\alpha}} G^{z-\alpha}(\cdot - y)G^{z+\alpha}(y - \cdot) \otimes (\chi_-, \cdot)_{\mathbb{C}^2}\chi_+ \\ & + \frac{\mathrm{i}\hat{\beta}_-}{(4\pi)^2\hat{\beta}_+\hat{\beta}_- \sqrt{z-\alpha}\sqrt{z+\alpha}} G^{z+\alpha}(\cdot - y)G^{z-\alpha}(y - \cdot) \otimes (\chi_+, \cdot)_{\mathbb{C}^2}\chi_- + . \end{aligned} \tag{70}$$

The terms $(\chi_-, \cdot)_{\mathbb{C}^2}\chi_+$ and $(\chi_+, \cdot)_{\mathbb{C}^2}\chi_-$ indicate that, in such a case, the dynamics cannot be factorized in the spin components. Furthermore, there are not 'ready to use' formulae that can be used to evaluate the spectrum or the propagator of H^{od}.

4. Conclusions

In the previous sections, we introduced a family of Hamiltonians describing the dynamics of a quantum system consisting of one particle in interaction with an array of localized spins.

Different self-adjoint extensions of the free Hamiltonian correspond to different physical models of interaction between the particle and the spins. In fact it is possible to characterize particular sub-families of extensions according to different features of the dynamics they generate.

In example 1, we identified the sub-family of δ-like Hamiltonians. While the spin dynamics is unaffected by the interaction, the particle 'feels' zero-range forces whose strength depends on the value of some spin component of the localized spin. Those interaction models are a rigorous version of the spin-dependent delta potentials that have been one of the main tools in the description of neutron scattering by condensed matter [18].

Our current aim is to build up simple models for a quantum measurement apparatus detecting 'the trajectory' of a quantum particle. Mott first considered this problem in a seminal paper [21]. He was looking for an explanation of the appearance of sharp classical-like tracks in particle detectors in high energy physics experiments. Mott's paper remained almost unnoticed till the second half of the last century when a renewed interest in the measurement problem showed up in the community of theoretical physicists. Since that time the possibility of understanding at least some qualitative features of the measurement process thoroughly inside the framework of quantum mechanics, without relying on any 'reduction of the wave packet' postulate, has been matter of debate in fundamental and applied theoretical physics (see, e.g., [1, 2, 10, 11, 15, 16]).

The first attempt to analyse, in a simple setting, the dynamics of a quantum particle interacting with a many-body quantum system is due to Hepp ([14], see also [26] for recent results on the subject) . He defined a one-dimensional model (often referred to as the Coleman–Hepp model) of a quantum measurement apparatus suitable for the measure of the spin of a particle through its interaction with an array of localized spins. In order to simplify the treatment, the particle wavefunction was supposed to translate with constant velocity according to free non-dispersive dynamics. It is worth mentioning that the Hamiltonians described in example 1 might be used to define a completely quantum Coleman–Hepp model.

Following the original idea of Mott we started to analyse models similar to the one described by Hepp, where the dynamics of the spins is significantly affected by the particle wavefunction. The Hamiltonians described in example 2 make available a solvable model where rigorous results on the dynamics of a quantum particle in (a simplified version of) a particle detector might be obtained.

Acknowledgments

The authors thank Andrea Posilicano for useful exchanges of views about the theory of self-adjoint extensions of symmetric operators. We also thank Alessandro Teta and Gianfausto Dell'Antonio for all the enlightening discussions about Mott's paper. This work was supported by the EU-Project 'Quantum Probability with Applications to Physics, Information Theory and Biology'.

References

[1] Adami R, Figari R, Finco D and Teta A 2004 On the asymptotic behaviour of a quantum two-body system in the small mass ratio limit *J. Phys. A: Math. Gen.* **37** 7567–80

[2] Adami R, Finco D, Figari R and Teta A 2006 On the asymptotic dynamics of a quantum system composed by heavy and light particles *Commun. Math. Phys.* **268** 819–52

[3] Akhiezer N and Glazman I M 1981 *Theory of Linear Operators in Hilbert Space* vol II (Boston, MA: Pitman)

[4] Albeverio S, Gesztesy F, Høegh-Krohn R and Holden H 2005 *Solvable Models in Quantum Mechanics* 2nd edn (Providence, RI: American Mathematical Society) (with an Appendix by P. Exner)

[5] Albeverio S and Kurasov P 2000 *Singular Perturbations of Differential Operators: Solvable Schrödinger-Type Operators* (Cambridge: Cambridge University Press)

[6] Albeverio S and Pankrashkin K 2005 A remark on Krein's resolvent formula and boundary conditions *J. Phys. A: Math. Gen.* **38** 4859–64

[7] Avdonin S A, Dmitrieva L A, Kuperin Yu A and Sartan V V 2005 Solvable model of spin-dependent transport through a finite array of quantum dots *J. Phys. A: Math. Gen.* **38** 4825–33
[8] Berezin F A and Faddeev L D 1961 A remark on Schrödinger's equation with a singular potential *Soviet Math. Dokl.* **2** 372–5
[9] Bethe H and Peierls R 1935 Quantum theory of the diplon *Proc. R. Soc.* A **148** 146–56
[10] Cacciapuoti C, Carlone R and Figari R 2005 Decoherence induced by scattering: a three-dimensional model *J. Phys. A: Math. Gen.* **38** 4933–46
[11] Dürr D, Figari R and Teta A 2004 Decoherence in a two-particle model *J. Math. Phys.* **45** 1291–309
[12] Fermi E 1936 Sul moto dei neutroni nelle sostanze idrogenate *Ric. Sci.* **7** 13–52
Fermi E 1962 *E. Fermi Collected Papers, vol. I, Italy, 1921–1938* (Chicago, IL: University of Chicago Press) pp 980–1016 (Engl. Transl.)
[13] Gorbachuk V I and Gorbachuk M L 1991 *Boundary Value Problems for Operator Differential Equations* (Dordrecht: Kluwer)
[14] Hepp K 1972 Quantum theory of measurement and macroscopic observables *Helv. Phys. Acta* **45** 237–48
[15] Hornberger K and Sipe J E 2003 Collisional decoherence reexamined *Phys. Rev.* A **68** 012105
[16] Joos E and Zeh H D 1985 The emergence of classical properties through interaction with the environment *Z. Phys.* B **59** 223–43
[17] Kronig R de L and Penney W G 1931 Quantum mechanics of electrons in crystal lattices *Proc. R. Soc.* A **130** 499–513
[18] Lovesey S W 1984 *Theory of Neutron Scattering from Condensed Matter* vol I (Oxford: Clarendon)
[19] Šeba P, Exner P, Pichugin K N, Vyhnal A and Středa P 2001 Two-component interference effect: model of a spin-polarized transport *Phys. Rev. Lett.* **86** 1598–1601
[20] Minlos R A and Faddeev L D 1962 On the point interaction for a three-particle system in quantum mechanics *Dokl. Akad. Nauk SSSR* **141** 1335–8 (in Russian)
Minlos R A and Faddeev L D 1962 On the point interaction for a three-particle system in quantum mechanics *Soviet Phys. Dokl.* **6** 1072–4 (Translated)
[21] Mott N F 1929 The wave mechanics of α-ray tracks *Proc. R. Soc.* A **126** 79–84
[22] Pankrashkin K 2006 Resolvents of self-adjoint extensions with mixed boundary conditions *Rep. Math. Phys.* **58** 207–21
[23] Posilicano A 2001 A Kreĭn-like formula for singular perturbations of self-adjoint operators and applications *J. Funct. Anal.* **183** 109–47
[24] Reed M and Simon B 1972 *Methods of Modern Mathematical Physics I: Functional Analysis* (New York: Academic)
[25] Reed M and Simon B 1975 *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness* vol II (New York: Academic)
[26] Sewell G 2005 On the mathematical structure of quantum measurement theory *Rep. Math. Phys.* **56** 271–90
[27] Thomas L H 1935 The interaction between a neutron and a proton and the structure of H^3 *Phys. Rev.* **47** 903–9